Uitwerking Final Exam
Kwantumfysica 1   5 Feb 2008

1/2

Problem 1

The quantum state is given as a function of position, but we need to know the state in relation to velocity.

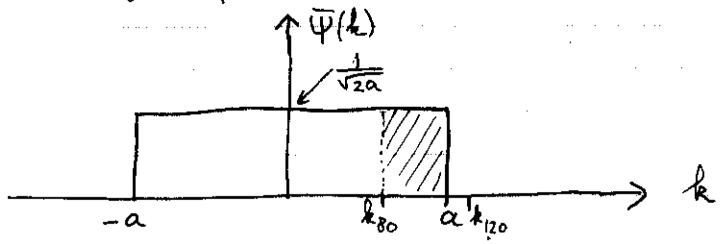
Velocity is proportional to momentum and k-number

$$v = \frac{p}{m} = \frac{\hbar k}{m}$$

We must therefore evaluate this state using the Fourier transform $\bar{\Psi}(k)$ of the state $\Psi(x)$

$$\bar{\Psi}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x)\, e^{-ikx}\,dx = \begin{cases} \frac{1}{\sqrt{2a}} & , |k| \le a \\ 0 , & |k| > a \end{cases},$$

where we used the standard Fourier transform on p. 1 of the problem set.



The probability for a measurement result between + 80 km/s and 120 km/s is now

2/2

$$P = \int_{k_{80}}^{k_{120}} \bar{\Psi}(k)^* \, \bar{\Psi}(k)\, dk \quad , \text{ where}$$

$k_{80} = \frac{v_{80} m_e}{\hbar}$ for $v_{80} = 80$ km/s and

$k_{120} = \frac{v_{120} m_e}{\hbar}$ for $v_{120} = 120$ km/s.

To evaluate this integral, we need to compare k_{80} and k_{120} to a (also sketched in figure, not to scale)

$$\left.\begin{aligned} a &= 10^9 \text{ m}^{-1} \\ k_{80} &= \frac{80 \cdot 10^3 \cdot 9.1 \cdot 10^{-31}}{1.055 \cdot 10^{-34}} = 0{,}690 \cdot 10^9 \text{ m}^{-1} \\ k_{120} &= \frac{120 \cdot 10^3 \cdot 9.1 \cdot 10^{-31}}{1.055 \cdot 10^{-34}} = 1{,}035 \cdot 10^9 \text{ m}^{-1} \end{aligned}\right\} k_{80} < a < k_{120} \Rightarrow$$

$$P = \int_{k_{80}}^{a} \left(\frac{1}{\sqrt{2a}}\right)^2 dk = \frac{1}{2a}(a - k_{80})$$

$$= \frac{1}{2}\left(\frac{1 - 0{,}69}{1}\right) = 0{,}155$$

So, probability is 15.5 %

Problem 2

3/12

a) $\hat{H}$ and $\hat{A}$ commute when $[\hat{H},\hat{A}] = \hat{H}\hat{A} - \hat{A}\hat{H} = 0 \Rightarrow$

$$\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix} - \begin{pmatrix} -a & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix} = \begin{pmatrix} -aE_0 & aT \\ -aT & aE_0 \end{pmatrix} - \begin{pmatrix} -aE_0 & -aT \\ aT & aE_0 \end{pmatrix} = \begin{pmatrix} 0 & +2aT \\ -2aT & 0 \end{pmatrix}$$

$\neq \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \Rightarrow \hat{H}$ and $\hat{A}$ do not commute

(We give here full derivation, but only proof, given the eigenstates was sufficient)

b) We need to solve the eigen value problem

$$\hat{H}|\varphi_i\rangle = E_i|\varphi_i\rangle \Rightarrow \begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = E_i\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \Rightarrow$$

For eigen values Solve $\begin{vmatrix} E_0 - E_i & T \\ T & E_0 - E_i \end{vmatrix} = 0 \Rightarrow$

$$(E_0 - E_i)^2 - T^2 = 0 \Rightarrow E_i^2 - 2E_0E_i + E_0^2 - T^2 = 0 \Rightarrow$$

$$E_i = \frac{2E_0 \pm \sqrt{4E_0^2 - 4(E_0^2 - T^2)}}{2} = E_0 \pm T$$

The eigen states that belong to these two eigen values:

$$\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (E_0 + T)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

solving for c_1 and c_2 gives $c_1 = c_2 \Rightarrow$ normalized eigenstate is

$e^{i\varphi}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ is eigen state for eigenvalue $E_0 + T$

where we can choose the global phase $\varphi = 0 \Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$

Similar for the eigen value: $E_0 - T$

$$\begin{pmatrix} E_0 & T \\ T & E_0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = (E_0 - T)\begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$$

Solve again for c_1 and $c_2 \Rightarrow$ (let's fully write it out this time)

$$\begin{cases} E_0c_1 + Tc_2 - E_0c_1 + Tc_2 = 0 \\ Tc_1 + E_0c_2 - E_0c_2 + Tc_2 = 0 \end{cases} \Rightarrow \begin{cases} T(c_1 + c_2) = 0 \\ T(c_1 + c_2) = 0 \end{cases} \Rightarrow$$

$c_1 = -c_2 \Rightarrow$ For normalized state and global phase that makes c_1 real and positive this gives $\begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

$T < 0$, therefore $(E_0 + T) < (E_0 - T) \Rightarrow$

ground state is $|\varphi_g\rangle$ with $E_g = E_0 + T$

excited state is $|\varphi_e\rangle$ with $E_e = E_0 - T$

c) The state that we need to calculate the probability for is $|\varphi_L\rangle$

c-i) $P = |\langle\varphi_L|\varphi_g\rangle|^2 = |\langle\varphi_L|(\frac{1}{\sqrt{2}}|\varphi_L\rangle + \frac{1}{\sqrt{2}}|\varphi_R\rangle)|^2$

$= |\frac{1}{\sqrt{2}} + 0|^2 = \frac{1}{2}$ OR in matrix notation

$$P = \left|\begin{pmatrix} 1 & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right|^2 = \frac{1}{2}$$

5/12

c-ii) $P = \left|(1 \;\; 0)\begin{pmatrix}\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\end{pmatrix}\right|^2 = \frac{1}{2}$

c-iii) $P = \left|(1 \;\; 0)\cdot\left(\begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix} + \begin{pmatrix}\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\end{pmatrix}\right)\frac{1}{\sqrt{2}}\right|^2 = \left|(1 \;\; 0)\begin{pmatrix}1 \\ 0\end{pmatrix}\right|^2 = 1$

c-iv) $P = \left|\langle \varphi_L | \left(\frac{1}{\sqrt{2}}|\varphi_g\rangle - \frac{1}{\sqrt{2}}|\varphi_e\rangle\right)\right|^2 = \left|\langle \varphi_L | \varphi_R\rangle\right|^2 = 0$

d) $\langle \varphi_g | \hat{A} | \varphi_g\rangle = \left(\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right)\begin{pmatrix}-a & 0 \\ 0 & a\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix} = 0$ $\Rightarrow$ expectation value for position is zero for system in state $|\varphi_g\rangle$

$\langle \varphi_e | \hat{A} | \varphi_e\rangle = \left(\frac{1}{\sqrt{2}} \;\; -\frac{1}{\sqrt{2}}\right)\begin{pmatrix}-a & 0 \\ 0 & a\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\end{pmatrix} = 0$ " " " $|\varphi_e\rangle$

$\langle \varphi_g | \hat{A} | \varphi_e\rangle = \left(\frac{1}{\sqrt{2}} \;\; \frac{1}{\sqrt{2}}\right)\begin{pmatrix}-a & 0 \\ 0 & a\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}}\end{pmatrix} = -a$

$\langle \varphi_e | \hat{A} | \varphi_g\rangle = \left(\frac{1}{\sqrt{2}} \;\; -\frac{1}{\sqrt{2}}\right)\begin{pmatrix}-a & 0 \\ 0 & a\end{pmatrix}\begin{pmatrix}\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}}\end{pmatrix} = -a$

When the system is in a superposition of $|\varphi_g\rangle$ and $|\varphi_e\rangle$ the expectation value for position can be different from zero.

e) The state at $t=0$ is denoted as $|\psi_0\rangle = |\varphi_L\rangle = \frac{1}{\sqrt{2}}(|\varphi_g\rangle + |\varphi_e\rangle)$ since the measurement result was "left" $= -a$.

For investigating time evolution of $\langle\hat{A}\rangle$, describe the state of the system as a superposition of energy eigen states:

$$\langle \hat{A}(t)\rangle = \langle \psi(t)|\hat{A}|\psi(t)\rangle = \langle\psi_0|\hat{U}^+\hat{A}\hat{U}|\psi_0\rangle$$

6/12

with $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t}$ $\Rightarrow$

$$\langle \hat{A}(t)\rangle = \frac{1}{2}\left(\langle\varphi_g| + \langle\varphi_e|\right)\hat{U}^+\hat{A}\hat{U}\left(|\varphi_g\rangle + |\varphi_e\rangle\right)$$

$$= \frac{1}{2}\left(e^{+i\omega_g t}\langle\varphi_g| + e^{+i\omega_e t}\langle\varphi_e|\right)\hat{A}\left(e^{-i\omega_g t}|\varphi_g\rangle + e^{-i\omega_e t}|\varphi_e\rangle\right)$$

(where $\omega_g = E_g/\hbar$ and $\omega_e = E_e/\hbar$)

$$= \frac{1}{2}\left(\langle\varphi_g|\hat{A}|\varphi_g\rangle + \langle\varphi_e|\hat{A}|\varphi_e\rangle + e^{+i(\omega_g-\omega_e)t}\langle\varphi_g|\hat{A}|\varphi_e\rangle + e^{+i(\omega_e-\omega_g)t}\langle\varphi_e|\hat{A}|\varphi_g\rangle\right)$$

$$= \frac{1}{2}\left(0 + 0 + e^{-i(\omega_e-\omega_g)t}(-a) + e^{+i(\omega_e-\omega_g)t}(-a)\right)$$

$$= -\frac{1}{2}a\, 2\cos\left((\omega_e-\omega_g)t\right)$$

$$= -a\cos\left((\omega_e-\omega_g)\,t\right) = -a\cos\left(\frac{2|T|}{\hbar}\cdot t\right)$$

The system oscillates between the two wells, and (as it should be) the dynamics starts at position $-a$ for $t=0$. The amplitude is a, so the particle goes from $-a$ to $+a$ and back and so forth.

The angular frequency is set by the strength of the tunnel coupling $|T|$, and equal to $\frac{2|T|}{\hbar}$.

7/12

Problem 3

a) a-i) With the result of problem 2c (or calculate directly, see problem 3d)

$P_{L1} = \frac{1}{2}$, $P_{L2} = \frac{1}{2}$, so both particles in left well has probability $P_{LL} = P_{L1} \cdot P_{L2} = \frac{1}{4}$

a-ii) As for a-i), $P_{L1} = \frac{1}{2}$, $P_{L2} = \frac{1}{2}$, so $P_{LL} = \frac{1}{4}$

b) b-i) Need to proof $\hat{H}|\Psi_T\rangle_{c\alpha} = (E_g + E_e)|\Psi_T\rangle_{c\alpha} \Rightarrow$

$$\hat{H}|\Psi_T\rangle_{c\alpha} = \hat{H}|\varphi_{g1}\rangle|\varphi_{e2}\rangle = (\hat{H}_1 + \hat{H}_2)|\varphi_{g1}\rangle|\varphi_{e2}\rangle$$
$$= (E_g|\varphi_{g1}\rangle)|\varphi_{e2}\rangle + |\varphi_{g1}\rangle(E_e|\varphi_{e2}\rangle)$$
$$= (E_g + E_e)|\varphi_{g1}\rangle|\varphi_{e2}\rangle = (E_g + E_e)|\Psi_T\rangle_{c\alpha} \quad \text{q.e.d.}$$

b-ii) As for b-i), need to proof $\hat{H}|\Psi_T\rangle_{c\beta} = (E_g + E_e)|\Psi_T\rangle_{c\beta} \Rightarrow$

$$\hat{H}|\Psi_T\rangle_{c\beta} = (\hat{H}_1 + \hat{H}_2)|\varphi_{e1}\rangle|\varphi_{g2}\rangle = E_e|\varphi_{e1}\rangle|\varphi_{g2}\rangle + |\varphi_{e1}\rangle E_g|\varphi_{g2}\rangle = (E_g + E_e)|\Psi_T\rangle_{c\beta}$$

b-iii) Need to proof $\hat{H}|\Psi_T\rangle_{\alpha\beta} = (E_g + E_e)|\Psi_T\rangle_{\alpha\beta} \Rightarrow$

$$\hat{H}|\Psi_T\rangle_{\alpha\beta} = (\hat{H}_1 + \hat{H}_2)(\alpha|\varphi_{g1}\rangle|\varphi_{e2}\rangle + \beta|\varphi_{e1}\rangle|\varphi_{g2}\rangle)$$
$$= \alpha((E_g + E_e)|\varphi_{g1}\rangle|\varphi_{e2}\rangle) + \beta((E_g + E_e)|\varphi_{e1}\rangle|\varphi_{g2}\rangle)$$
$$= (E_g + E_e)(\alpha|\Psi_T\rangle_{c\alpha} + \beta|\Psi_T\rangle_{\beta}) = (E_g + E_e)|\Psi_T\rangle_{\alpha\beta} \quad \underline{\text{q.e.d.}}$$

0/12

c) Exchanging particles means putting all cases of particle 1 in state (or orbital) $|\varphi_{g1}\rangle$ into $|\varphi_{e2}\rangle$ (so it becomes $|\varphi_{e1}\rangle$) and $|\varphi_{e1}\rangle$ into $|\varphi_{g2}\rangle$, and vice versa for particle 2

$$|\Psi_T\rangle_S = \frac{1}{\sqrt{2}}|\varphi_{g1}\rangle|\varphi_{e2}\rangle + \frac{1}{\sqrt{2}}|\varphi_{e1}\rangle|\varphi_{g2}\rangle \xleftrightarrow{\text{exchange}}$$
$$\frac{1}{\sqrt{2}}|\varphi_{e1}\rangle|\varphi_{g2}\rangle + \frac{1}{\sqrt{2}}|\varphi_{g1}\rangle|\varphi_{e2}\rangle =$$
$$\frac{1}{\sqrt{2}}|\varphi_{g1}\rangle|\varphi_{e2}\rangle + \frac{1}{\sqrt{2}}|\varphi_{e1}\rangle|\varphi_{g2}\rangle = +|\Psi_T\rangle_S$$

So $|\Psi_T\rangle_S$ is symmetric under exchange of particles

$$|\Psi_T\rangle_{AS} = \frac{1}{\sqrt{2}}|\varphi_{g1}\rangle|\varphi_{e2}\rangle - \frac{1}{\sqrt{2}}|\varphi_{e1}\rangle|\varphi_{g2}\rangle \xleftrightarrow{\text{exchange}}$$
$$\frac{1}{\sqrt{2}}|\varphi_{e1}\rangle|\varphi_{g2}\rangle - \frac{1}{\sqrt{2}}|\varphi_{g1}\rangle|\varphi_{e2}\rangle =$$
$$-\frac{1}{\sqrt{2}}|\varphi_{g1}\rangle|\varphi_{e2}\rangle + \frac{1}{\sqrt{2}}|\varphi_{e1}\rangle|\varphi_{g2}\rangle = -|\Psi_T\rangle_{AS}$$

So $|\Psi_T\rangle_{AS}$ is anti-symmetric under exchange of two identical particles.

d) Probability P_{LL} for both in the left well at $-a$:

d-i) $$P_{LL} = \left| \langle\varphi_{L1}|\langle\varphi_{L2}|\left(|\Psi_T\rangle_S\right)\right|^2$$

$$= \left|\langle\varphi_{L1}|\langle\varphi_{L2}|\left(\tfrac{1}{\sqrt{2}}|\varphi_{g1}\rangle|\varphi_{e2}\rangle + \tfrac{1}{\sqrt{2}}|\varphi_{e1}\rangle|\varphi_{g2}\rangle\right)\right|^2$$

$$= \left|\langle\varphi_{L1}|\langle\varphi_{L2}|\left(\tfrac{1}{2\sqrt{2}}(|\varphi_{L1}\rangle + |\varphi_{R1}\rangle)(|\varphi_{L2}\rangle - |\varphi_{R2}\rangle) + \tfrac{1}{2\sqrt{2}}(|\varphi_{L1}\rangle - |\varphi_{R1}\rangle)(|\varphi_{L2}\rangle + |\varphi_{R2}\rangle)\right)\right|^2$$

$$= \Big|\langle\varphi_{L1}|\langle\varphi_{L2}|\Big(\tfrac{1}{2\sqrt{2}}\big(|\varphi_{L1}\rangle|\varphi_{L2}\rangle - |\varphi_{L1}\rangle|\varphi_{R2}\rangle + |\varphi_{R1}\rangle|\varphi_{L2}\rangle - |\varphi_{R1}\rangle|\varphi_{R2}\rangle$$
$$+ |\varphi_{L1}\rangle|\varphi_{L2}\rangle + |\varphi_{L1}\rangle|\varphi_{R2}\rangle - |\varphi_{R1}\rangle|\varphi_{L2}\rangle - |\varphi_{R1}\rangle|\varphi_{R2}\rangle\big)\Big)\Big|^2$$

$$= \left|\tfrac{1}{2\sqrt{2}}\left(1 - 0 + 0 - 0 + 1 + 0 - 0 - 0\right)\right|^2 = \left(\frac{2}{2\sqrt{2}}\right)^2 = \frac{1}{2}$$

d-ii) $$P_{LL} = \left|\langle\varphi_{L1}|\langle\varphi_{L2}|\left(|\Psi_T\rangle_{AS}\right)\right|^2$$

$$= \left|\langle\varphi_{L1}|\langle\varphi_{L2}|\left(\tfrac{1}{\sqrt{2}}|\varphi_{g1}\rangle|\varphi_{e2}\rangle + \tfrac{1}{\sqrt{2}}|\varphi_{e1}\rangle|\varphi_{g2}\rangle\right)\right|^2$$

$$= \Big|\langle\varphi_{L1}|\langle\varphi_{L2}|\Big(\tfrac{1}{2\sqrt{2}}\big(|\varphi_{L1}\rangle|\varphi_{L2}\rangle - |\varphi_{L1}\rangle|\varphi_{R2}\rangle + |\varphi_{R1}\rangle|\varphi_{L2}\rangle - |\varphi_{R1}\rangle|\varphi_{R2}\rangle$$
$$- |\varphi_{L1}\rangle|\varphi_{L2}\rangle - |\varphi_{L1}\rangle|\varphi_{R2}\rangle + |\varphi_{R1}\rangle|\varphi_{L2}\rangle - |\varphi_{R1}\rangle|\varphi_{R2}\rangle\big)\Big)\Big|^2$$

$$= \left|\tfrac{1}{2\sqrt{2}}\left(1 - 0 + 0 - 0 - 1 - 0 + 0 + 0\right)\right|^2 = 0$$

Problem 4

a) $\hat{H} = \hat{T} + \hat{V}$ (kinetic + potential energy)

In x-representation, with constants used being m, ω_0 ⇒

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega_0^2 x^2$$

b) $\int_{-\infty}^{\infty} \psi^*(x)\,\psi(x)\,dx = 1$ when normalized ⇒

$$\int_{-\infty}^{\infty} A^2 e^{-2bx^2}dx = 1 \Rightarrow A^2\int_{-\infty}^{\infty} e^{-(\sqrt{2b}x)^2}\frac{1}{\sqrt{2b}}\,d(\sqrt{2b}x) = 1 \Rightarrow$$

$$A^2\frac{1}{\sqrt{2b}}\sqrt{\pi} = 1 \Rightarrow A = \left(\frac{2b}{\pi}\right)^{1/4}$$

c) Say $|\psi\rangle = \sum_{n=0}^{\infty} c_n|\chi_n\rangle$ ⇒

$$\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle} = \frac{\sum_n E_n|c_n|^2}{\sum_n |c_n|^2} > \frac{\sum_n E_0|c_n|^2}{\sum_n |c_n|^2}$$

$$= E_0\frac{\sum_n |c_n|^2}{\sum_n |c_n|^2} = E_0 \qquad \text{q.e.d.}$$

since all $E_n > E_0$ for $n > 0$

d) We need to minimize $\frac{\langle\psi|\hat{H}|\psi\rangle}{\langle\psi|\psi\rangle}$ under variation of b.

Note that $\langle\psi|\psi\rangle$ always equals 1 if we always use $A=\left(\frac{2b}{\pi}\right)^{1/4}$ (from question b)), so we only need to minimize $\langle\psi|\hat{H}|\psi\rangle$ in that case, that is solve $\frac{d(\langle\psi|\hat{H}|\psi\rangle)}{db}=0$.

$$\langle\psi|\hat{H}|\psi\rangle = \langle\psi|\hat{T}|\psi\rangle + \langle\psi|\hat{V}|\psi\rangle .$$

$$\langle\psi|\hat{V}|\psi\rangle = \sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} e^{-bx^2}\left(\tfrac{1}{2}m\omega_0^2x^2\right)e^{-bx^2}\,dx$$

$$= \sqrt{\frac{2b}{\pi}}\,\frac{\frac{1}{2}m\omega_0^2}{2b}\int_{-\infty}^{\infty} 2bx^2\, e^{-2bx^2}\frac{1}{\sqrt{2b}}\,d(\sqrt{2b}\,x)$$

$$= \sqrt{\frac{2b}{\pi}}\,\frac{\frac{1}{2}m\omega_0^2}{2b}\,\frac{1}{\sqrt{2b}}\cdot\tfrac{1}{2}\sqrt{\pi} = \frac{m\omega_0^2}{8b}$$

$$\langle\psi|\hat{T}|\psi\rangle = \sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} e^{-bx^2}\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\right)e^{-bx^2}\,dx$$

$$= -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} e^{-bx^2}\left(-2be^{-bx^2}+4b^2x^2e^{-bx^2}\right)dx$$

$$= -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\int_{-\infty}^{\infty} -2b\,e^{-2bx^2} + 4b^2x^2e^{-2bx^2}\,dx$$

$$= -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\left(-2b\int_{-\infty}^{\infty} e^{-2bx^2}\frac{1}{\sqrt{2b}}\,d(\sqrt{2b}\,x) + 2b\int_{-\infty}^{\infty}(\sqrt{2b}\,x)^2 e^{-(\sqrt{2b}\,x)^2}\frac{1}{\sqrt{2b}}\,d(\sqrt{2b}\,x)\right)$$

$$= -\frac{\hbar^2}{2m}\sqrt{\frac{2b}{\pi}}\left(\frac{-2b\sqrt{\pi}}{\sqrt{2b}} + \frac{2b\,\frac{1}{2}\sqrt{\pi}}{\sqrt{2b}}\right) = \frac{\hbar^2 b}{2m}$$

$$\Rightarrow \langle\psi|\hat{H}|\psi\rangle = \frac{m\omega_0^2}{8b} + \frac{\hbar^2 b}{2m} \Rightarrow$$

$$\frac{d(\langle\psi|\hat{H}|\psi\rangle)}{db} = \frac{d}{db}\left(\frac{m\omega_0^2}{8b}+\frac{\hbar^2 b}{2m}\right) = \frac{\hbar^2}{2m} - \frac{m\omega_0^2}{8b^2} = 0 \Rightarrow$$

$$\omega_0^2 = \frac{8\hbar^2 b^2}{2m} \Rightarrow b = \frac{m\omega_0}{2\hbar} \Rightarrow$$

$$\langle\psi|\hat{T}|\psi\rangle = \frac{\hbar^2 b}{2m} = \frac{1}{4}\hbar\omega_0$$

$$\langle\psi|\hat{V}|\psi\rangle = \frac{m\omega_0^2}{8b} = \frac{1}{4}\hbar\omega_0$$

$$\langle\psi|\hat{H}|\psi\rangle = E_0 = \frac{1}{2}\hbar\omega_0$$ (agrees indeed with harmonic oscillator ground state)

$$A = \left(\frac{2b}{\pi}\right)^{1/4} = \left(\frac{m\omega_0}{\pi\hbar}\right)^{1/4}$$

e) $\langle T\rangle = \frac{1}{4}\hbar\omega_0$, $\langle\hat{V}\rangle = \frac{1}{4}\hbar\omega_0$ (see d))

Heisenberg states $\Delta x\,\Delta p \geq \hbar/2$, so if the particle was truly at the bottom of the well, this would give $\langle\hat{V}\rangle=0$ with $\Delta x=0$. Then, Δp must be very high, so $\langle\hat{T}\rangle$ very high and this high energy cost for $\langle\hat{T}\rangle$ makes that it is not the ground state. Instead, a trade off with both $\langle T\rangle$ and $\langle\hat{V}\rangle$ a bit more than zero gives a state with minimal energy.